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1983 J. Phys. A: Math. Gen. 16 L27

(http://iopscience.iop.org/0305-4470/16/1/006)

J. Phys. A: Math. Gen. 16 (1983) L27–L29. Printed in Great Britain

LETTER TO THE EDITOR

Quantum electrodynamic formulation of the Josephson tunnelling theory

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Received 24 August 1982

Abstract. The local ground state polarisation (complex) Schwinger Lagrangian is given corresponding to the Josephson theory of tunnelling in a junction capacitor. In the quantum electrodynamic theory there are two regimes of interest: (i) for voltages small on the scale of the gap $|eV| < 2\Delta$, the motion is that of a 'quantum pendulum' with no dissipative damping; (ii) for voltages large on the scale of the gap $|eV| > 2\Delta$, a finite shunt conductance is 'turned on' describing the dielectric breakdown of the junction capacitor.

In a classic paper on vacuum polarisation (Schwinger 1951) the rules were given for constructing the nonlinear electromagnetic local Lagrangian obtained after ground state averaging over electronic degrees of freedom. In a quantum electrodynamic circuit element (Widom 1979) the local Schwinger Lagrangian describes the electrodynamic degrees of freedom via the flux coordinate Φ and the Faraday law voltage across the element

$$V = -(\mathrm{d}\Phi/c\ \mathrm{d}t). \tag{1}$$

The Schwinger (ground state) Lagrangian has the complex form

$$\mathscr{L}(V, \Phi) = L(-cV, \Phi) + (\mathrm{i}\hbar/2)\Gamma(V, \Phi), \tag{2}$$

where the real part $L(\dot{\Phi}, \Phi)$ determines quantum interference of amplitudes (as does any other real Lagrangian), while Γ describes the transition rate per unit time of achieving a real excitation of the electronic degrees of freedom.

Our purpose is to discuss the physical meaning of the Schwinger Lagrangian for the special case of a tunnelling junction in the Josephson theory, i.e. when the action is computed to second order in the one-electron tunnelling amplitudes assumed real (Ambegaokar *et al* 1982). We differ from Ambegaokar *et al* in one very important respect to be discussed in what follows (Werthamer 1966).

The Schwinger Lagrangian corresponding to the Josephson theory is given by

$$\mathscr{L}(V, \Phi) = \tfrac{1}{2}\varepsilon(\bar{\omega}_V + \mathrm{i}0^+)CV^2 + \hbar\nu\cos(2e\Phi/\hbar c), \tag{3}$$

where ν is the electron pair tunnelling frequency, C is the geometrical junction capacitance,

$$\bar{\omega}_V = (eV/\hbar) \tag{4}$$

and $\varepsilon(\zeta)$ is the normal current dielectric response function whose physical significance is as follows. In an engineering circuit picture (Likharev 1979) the geometrical capacitance C is shunted by a normal current channel admittance $Y_n(\zeta)$. Adding admittances in parallel yields the effective capacitance $C_{\text{eff}}(\zeta)$ such that

$$-i\zeta C_{\text{eff}}(\zeta) = -i\zeta C + Y_n(\zeta), \qquad C_{\text{eff}}(\zeta) = \varepsilon(\zeta)C. \tag{5a, b}$$

Since normal current excitations can be real only if $\hbar\omega > 2\Delta$, and are certainly 'virtual' if $\hbar\omega < 2\Delta$, it follows that the normal current channel admittance obeys

$$\text{Re } Y_n(\omega + i0^+) = 0, \qquad \hbar\omega < 2\Delta, \tag{6}$$

where Δ is the superconducting gap.

Theorem 1. If the voltage across the junction is small on the scale of the superconducting gap Δ, then the transition rate for dissipative electronic excitation vanishes, i.e.

$$\Gamma(|eV| < 2\Delta) = 0. \tag{7}$$

Proof. From equations (2), (3) and (5), the power dissipated in the junction is

$$\hbar\bar{\omega}_V \Gamma(V) = V^2 \text{ Re } Y_n(\bar{\omega}_V + i0^+). \tag{8}$$

Hence, equation (7) holds by virtue of equations (6) and (8).

When described in terms of the normal shunt conductance

$$G(V) = \text{Re } Y_n(\bar{\omega}_V + i0^+), \tag{9}$$

one notes that $G(V)$ 'shuts off' when $|V| < (2\Delta/e)$ and 'turns on' when $|V| > (2\Delta/e)$. A simplified resistively shunted junction model holds that $G(V)$ is independent of voltage (Leggett 1980).

Ambegaokar *et al* show that a form of resistively shunted junction model can arise from an approximation to the action. However, the fact that $G(V)$ can turn on and off, as described above, appears to us to invalidate their approximation scheme as regards laboratory use of the Josephson theory.

Theorem 2. For flux paths $\Phi(t)$ such that $|\dot{\Phi}(t)| < 2c\Delta/e$, the amplitudes are weighted by a real non-dissipative Lagrangian path integral

$$\int D\Phi \exp(i/\hbar) \int L(\dot{\Phi}, \Phi)\, dt. \tag{10}$$

Proof. Equation (10) follows directly from equations (1), (2), (3) and (7).

Hence, when the voltage is small on the scale of the superconducting gap, the macroscopic Schrödinger equation

$$i\hbar\, \partial\psi(\Phi, t)/\partial t = [E(Q = i\hbar c\, \partial/\partial\Phi) - \hbar\nu \cos(2e\Phi/\hbar c)]\psi(\Phi, t) \tag{11}$$

provides an adequate description of junction dynamics. In equation (11), $E(Q)$ is defined so that

$$V = dE(Q)/dQ \tag{12a}$$

and

$$Q = (d/dV)[\tfrac{1}{2}CV^2 - (\hbar V/2e) \text{ Im } Y_n(\bar{\omega}_V + i0^+)] \tag{12b}$$

yield the same equation of state for the junction capacitor 'charge–voltage' relation, when the shunt conductance is 'turned off', i.e. when there is no dielectric breakdown of the capacitor.

Let us now compare the consequences of Schwinger's Lagrangian for vacuum polarisation with the consequences of the quantum electrodynamic formulation of the Josephson theory. When a few electrons in the vacuum produce electric fields too weak for vacuum dielectric breakdown, i.e. the production of real electron–positron pairs, then a simple Schrödinger equation is an adequate treatment of dynamics. Similarly, if the electrons in a tunnelling junction produce voltages too weak to induce capacitor dielectric breakdown, i.e. the production of real dissipative normal currents, then a simple Schrödinger equation (11) provides an adequate description.

Finally, for such weak voltages $Q \simeq \bar{C}V$; this implies for the Heisenberg equation of motion (corresponding to equation (11)) that

$$\ddot{\theta} + \omega_{\mathrm{p}}^2 \sin\theta = 0, \tag{13}$$

where $\theta = 2e\Phi/\hbar c$ and $\omega_{\mathrm{p}}^2 = 4e^2\nu/\hbar\bar{C}$. Thus, the simple Josephson pendulum is recovered as would be expected.

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